

### 3.7 XFEM CRACK TIP ENRICHMENTS

In this section, a brief review of major available tip enrichments is presented.

#### 3.7.1 Isotropic enrichment

Enrichment functions (3.10) are the basic and mostly used crack tip enrichment functions,

$$\{F_I(r, \theta)\}_{I=1}^4 = \left\{ \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \sin \theta \cos \frac{\theta}{2}, \sqrt{r} \sin \theta \sin \frac{\theta}{2} \right\} \quad (3.95)$$

It is important to note that the term  $\sqrt{r} \sin \theta / 2$  is discontinuous along the crack. As a result, once the enrichments (3.95) are adopted for the crack tip element, the discontinuous field can be automatically represented and there is no need for inclusion of the Heaviside part of the approximation.

Despite being obtained from an asymptotic crack tip analysis in a single isotropic homogeneous medium under static conditions for a traction-free crack, it has been frequently used for enhancement of the analyses in a wide variety of problems including orthotropic materials, non-homogeneous FGMs, hydraulic fracture, dynamic problems, bimetals, elastoplastic solutions, electromechanical problems, etc.

#### 3.7.2 Orthotropic enrichment functions

General crack tip enrichment functions presented in Assadpoure and Mohammadi (2007) in the local crack tip polar coordinate system  $(r, \theta)$  are defined as,

$$\{F_I(r, \theta)\}_{I=1}^4 = \left\{ \sqrt{r} \cos \frac{\theta_1}{2} \sqrt{g_1(\theta)}, \sqrt{r} \cos \frac{\theta_2}{2} \sqrt{g_2(\theta)}, \sqrt{r} \sin \frac{\theta_1}{2} \sqrt{g_1(\theta)}, \sqrt{r} \sin \frac{\theta_2}{2} \sqrt{g_2(\theta)} \right\} \quad (3.96)$$

where functions  $g_k(\theta)$  and  $\theta_k$ , in their general form for all orthotropic composites, are defined as:

$$g_k(\theta) = \sqrt{(\cos \theta + s_{kx} \sin \theta)^2 + (s_{ky} \sin \theta)^2} \quad (3.97)$$

$$\theta_k = \arctan \left( \frac{s_{ky} \sin \theta}{\cos \theta + s_{kx} \sin \theta} \right) \quad (3.98)$$

where  $s_{kx}$  and  $s_{ky}$  have been defined in Eq. (4.113).

#### 3.7.3 Bimaterial enrichments

Sukumar *et al.* (2004) developed the following enrichment functions from the analytical solution near an interface crack tip in isotropic bimetals,

$$\begin{aligned} \{F_l(r, \theta)\}_{l=1}^{12} = & \left\{ \sqrt{r} \cos(\varepsilon \ln r) e^{-\varepsilon \theta} \sin \frac{\theta}{2}, \sqrt{r} \cos(\varepsilon \ln r) e^{-\varepsilon \theta} \cos \frac{\theta}{2}, \right. \\ & \sqrt{r} \cos(\varepsilon \ln r) e^{\varepsilon \theta} \sin \frac{\theta}{2}, \sqrt{r} \cos(\varepsilon \ln r) e^{\varepsilon \theta} \cos \frac{\theta}{2}, \\ & \sqrt{r} \sin(\varepsilon \ln r) e^{-\varepsilon \theta} \sin \frac{\theta}{2}, \sqrt{r} \sin(\varepsilon \ln r) e^{-\varepsilon \theta} \cos \frac{\theta}{2}, \\ & \sqrt{r} \sin(\varepsilon \ln r) e^{\varepsilon \theta} \sin \frac{\theta}{2}, \sqrt{r} \sin(\varepsilon \ln r) e^{\varepsilon \theta} \cos \frac{\theta}{2}, \\ & \sqrt{r} \cos(\varepsilon \ln r) e^{\varepsilon \theta} \sin \frac{\theta}{2} \sin \theta, \sqrt{r} \cos(\varepsilon \ln r) e^{\varepsilon \theta} \cos \frac{\theta}{2} \sin \theta, \\ & \left. \sqrt{r} \sin(\varepsilon \ln r) e^{\varepsilon \theta} \sin \frac{\theta}{2} \sin \theta, \sqrt{r} \sin(\varepsilon \ln r) e^{\varepsilon \theta} \cos \frac{\theta}{2} \sin \theta \right\} \end{aligned} \quad (3.99)$$

$\varepsilon$  is the index of oscillation defined as

$$\varepsilon = \frac{1}{2\pi} \ln \left( \frac{1-\beta}{1+\beta} \right) \quad (3.100)$$

where  $\beta$  is the second Dundurs parameter for isotropic materials (Dundurs, 1969)

$$\beta = \frac{\mu_1(\kappa_2 - 1) - \mu_2(\kappa_1 - 1)}{\mu_1(\kappa_2 + 1) + \mu_2(\kappa_1 + 1)} \quad (3.101)$$

### 3.7.4 Orthotropic bimaterial enrichments

The analytical solutions (7.17) and (7.18) can be used to derive the necessary enrichment functions for orthotropic bimaterial interface crack tip (Esna Ashari and Mohammadi, 2011a)

$$\begin{aligned} \{F_l(r, \theta)\}_{l=1}^8 = & \left[ e^{-\varepsilon \theta_l} \cos \left( \varepsilon \ln(r_l) + \frac{\theta_l}{2} \right) \sqrt{r_l}, e^{-\varepsilon \theta_l} \sin \left( \varepsilon \ln(r_l) + \frac{\theta_l}{2} \right) \sqrt{r_l}, \right. \\ & e^{\varepsilon \theta_l} \cos \left( \varepsilon \ln(r_l) - \frac{\theta_l}{2} \right) \sqrt{r_l}, e^{\varepsilon \theta_l} \sin \left( \varepsilon \ln(r_l) - \frac{\theta_l}{2} \right) \sqrt{r_l}, \\ & e^{-\varepsilon \theta_s} \cos \left( \varepsilon \ln(r_s) + \frac{\theta_s}{2} \right) \sqrt{r_s}, e^{-\varepsilon \theta_s} \sin \left( \varepsilon \ln(r_s) + \frac{\theta_s}{2} \right) \sqrt{r_s}, \\ & \left. e^{\varepsilon \theta_s} \cos \left( \varepsilon \ln(r_s) - \frac{\theta_s}{2} \right) \sqrt{r_s}, e^{\varepsilon \theta_s} \sin \left( \varepsilon \ln(r_s) - \frac{\theta_s}{2} \right) \sqrt{r_s} \right] \end{aligned} \quad (3.102)$$

with

$$r_j = r \sqrt{\cos^2 \theta + Z_j^2 \sin^2 \theta}, \quad j = l, s, \quad \begin{cases} Z_l = p \\ Z_s = q \end{cases} \quad (3.103)$$

$$\theta_j = \tan^{-1}(Z_j \tan \theta), \quad j = l, s, \quad \begin{cases} Z_l = p \\ Z_s = q \end{cases} \quad (3.104)$$

where  $\varepsilon$  is the index of oscillation defined in (7.22) in terms of the second Dundurs parameter  $\beta$  for orthotropic materials and all other parameters have been defined in Section 7.2.2.

### 3.7.5 Dynamic enrichment

Belytschko and Chen (2004) used the analytical asymptotic solutions near a propagating crack in an isotropic medium to develop the dynamic enrichment functions. A simplified form of these functions can be written as,

$$\{F_l(r, \theta)\}_{l=1}^4 = \left\{ r_d \cos \frac{\theta_d}{2}, r_s \cos \frac{\theta_s}{2}, r_d \sin \frac{\theta_d}{2}, r_s \sin \frac{\theta_s}{2} \right\} \quad (3.105)$$

where

$$r_d^2 = x^2 + \beta_d^2 y^2 \quad (3.106)$$

$$r_s^2 = x^2 + \beta_s^2 y^2 \quad (3.107)$$

$$\tan \theta_d = \beta_d \tan \theta \quad (3.108)$$

$$\tan \theta_s = \beta_s \tan \theta \quad (3.109)$$

with

$$\beta_d^2 = 1 - \frac{v_c^2}{c_d^2} \quad (3.110)$$

$$\beta_s^2 = 1 - \frac{v_c^2}{c_s^2} \quad (3.111)$$

where  $v_c$  is the crack tip velocity and  $c_d$  and  $c_s$  are the dilatational and shear wave speeds, respectively,

$$c_d = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad (3.112)$$

$$c_s = \sqrt{\frac{\mu}{\rho}} \quad (3.113)$$

and  $\mu$  and  $\lambda$  are the Lamé coefficients and  $\rho$  is the material density.

Alternatively, Kabiri (2009) proposed the following enrichment functions for dynamic problems based on a simple idea of adding the influence of the crack tip velocity on classical isotropic enrichment functions,

$$\{F_l(r, \theta)\}_{l=1}^4 = \left\{ \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \sin \theta \sin \frac{\theta}{2}, \sqrt{r} \cos \theta \sin \frac{\theta}{2} \right\} \sqrt{\frac{\gamma_d^2 + \gamma_s^2}{2}} \quad (3.114)$$

with

$$\gamma_d^2 = 1 - \frac{v_c^2}{c_d^2} \sin^2 \theta \quad (3.115)$$

$$\gamma_s^2 = 1 - \frac{v_c^2}{c_s^2} \sin^2 \theta \quad (3.116)$$

### 3.7.6 Orthotropic dynamic enrichments for moving cracks

The following dynamic orthotropic enrichment functions can be derived for moving crack tip problems: (Motamedi and Mohammadi, 2012)

$$\{F_l(r, \theta)\}_{l=1}^4 = \left\{ \sqrt{r} \cos \frac{\theta_1}{2} \sqrt{g_1(\theta)}, \sqrt{r} \cos \frac{\theta_2}{2} \sqrt{g_2(\theta)}, \sqrt{r} \sin \frac{\theta_1}{2} \sqrt{g_1(\theta)}, \sqrt{r} \sin \frac{\theta_2}{2} \sqrt{g_2(\theta)} \right\} \quad (3.117)$$

where  $g_k(\theta)$  and  $\theta_k$ , ( $k=1,2$ ) are defined as

$$g_k(\theta) = \sqrt{(\cos \theta + m_{kx} \sin \theta)^2 + (m_{ky} \sin \theta)^2}, \quad k=1, 2 \quad (3.118)$$

$$\theta_k = \arctg \left( \frac{m_{ky} \sin \theta}{\cos \theta + m_{kx} \sin \theta} \right), \quad k=1, 2 \quad (3.119)$$

and  $(r, \theta)$  are considered from the crack tip position, as depicted in Fig. 3.1.  $m_{kx}$  and  $m_{ky}$  are the real and imaginary parts of  $m_k$  (roots of Eq. (5.36)):

$$m_{kx} = \text{Re}(m_k), \quad k=1, 2 \quad (3.120)$$

$$m_{ky} = \text{Im}(m_k), k=1, 2 \quad (3.121)$$

### 3.7.7 Bending plates

For crack problems in bending plates and shells, the displacement fields around the crack tip can be decomposed to in-plane and out of plane components. Accordingly, a different set of enrichment functions are assigned to each component. For in-plane deformation, the enrichment functions are:

$$\{F_l(r, \theta)\}_{l=1}^4 = \left\{ \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \sin \theta \cos \frac{\theta}{2}, \sqrt{r} \sin \theta \sin \frac{\theta}{2} \right\} \quad (3.122)$$

The out of plane and rotational enrichments can be defined as (Bayesteh and Mohammadi, 2011),

$$\{F_l(r, \theta)\}_{l=1}^4 = \left\{ \sqrt{r} \sin \frac{\theta}{2}, r^{3/2} \sin \frac{\theta}{2}, r^{3/2} \cos \frac{\theta}{2}, r^{3/2} \sin \frac{3\theta}{2}, r^{3/2} \cos \frac{3\theta}{2} \right\} \quad (3.123)$$

$$\{F_l(r, \theta)\}_{l=1}^4 = \left\{ \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \sin \theta \cos \frac{\theta}{2}, \sqrt{r} \sin \theta \sin \frac{\theta}{2} \right\} \quad (3.124)$$

### 3.7.8 Crack tip enrichments in shells

Folias (1999) has illustrated that a correlation function, independent of angle or distance from the crack tip, exists between plates and shells:

$$\sigma_{shell} \approx \sigma_{plate} \left\{ 1 + \left[ c_1 + c_2 \left( \ln \frac{c}{\sqrt{Rh}} \right) + O\left( \frac{1}{R^2} \right) \right] \right\}, c_1, c_2, c_3 = \text{constant} \quad (3.125)$$

Similarly, Zehnder and Viz (2005) have stated that: “Near the crack tip, the stress distribution in shells is the same as that of plates and the shape of the shell or plate will come in only through the stress intensity factors”. Accordingly, similar enrichment functions can be used for approximation of the displacement field around a crack tip (Bayesteh and Mohammadi, 2011).

### 3.7.9 Electro-mechanical enrichment

Beginning with the constitutive equation in an electro-mechanical plane strain state (Fig 3.19), the mechanical strain tensor  $\varepsilon_{ij}$  and the electric field  $E_i$  can be defined in terms of the Cauchy stress tensor  $\sigma_{ij}$  and the electric displacement vector  $D_i$  (Sharma *et al.*, 2012),

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ E_x \\ E_y \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 & 0 & b_{21} \\ c_{12} & c_{22} & 0 & 0 & b_{22} \\ 0 & 0 & c_{33} & b_{13} & 0 \\ 0 & 0 & -b_{13} & \delta_{11} & 0 \\ -b_{21} & -b_{22} & 0 & 0 & \delta_{22} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ D_x \\ D_y \end{Bmatrix} \quad (3.126)$$

where  $c_{ij}$ ,  $b_{ij}$  and  $\delta_{ij}$  are mechanical and electrical material constants.

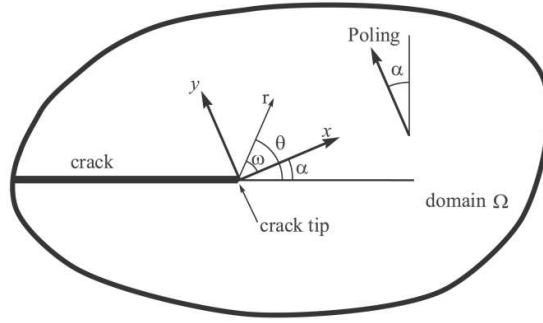


**Figure 3.19** Material axes at the crack tip and its polarization in an infinite domain.

The characteristic equation can be written in terms of  $s = s_1 + is_2$  (Sharma *et al.*, 2012),

$$\begin{aligned} & c_{11}\delta_{11}s^6 + \{c_{11}\delta_{22} + (2c_{12} + c_{33})\delta_{11} + b_{12}(b_{12} + 2b_{13}) + b_{13}^2\}s^4 + \\ & \{c_{22}\delta_{11} + (2c_{12} + c_{33})\delta_{22} + 2b_{22}(b_{12} + b_{13})\}s^2 + \{c_{22}\delta_{22} + b_{22}^2\} = 0 \end{aligned} \quad (3.127)$$

with six roots  $s_m = s_{m1} + is_{m2}$ . XFEM crack tip enrichment functions for the mechanical displacement  $\mathbf{u}$  and electric potential  $\varphi$  can be written as (Bechet *et al.*, 2009 and Sharma *et al.*, 2012),

$$\{F_l(r, \theta, \mu_k)\}_{l=1}^6 = \{\sqrt{r}g_1(\theta), \sqrt{r}g_2(\theta), \sqrt{r}g_3(\theta), \sqrt{r}g_4(\theta), \sqrt{r}g_5(\theta), \sqrt{r}g_6(\theta)\} \quad (3.128)$$

where

$$g_m(\theta) = \begin{cases} \rho_m(\omega, s_m) \cos\left(\frac{\psi(\omega, s_m)}{2}\right) & s_{m2} > 0 \\ \rho_m(\omega, s_m) \sin\left(\frac{\psi(\omega, s_m)}{2}\right) & s_{m2} < 0 \end{cases} \quad (3.129)$$

$$\psi(\omega, s_m) = \frac{\pi}{2} + \pi \operatorname{int}\left(\frac{\omega}{\pi}\right) - \tan^{-1} \left( \frac{\cos\left[\theta - \pi \operatorname{int}\left(\frac{\omega}{\pi}\right)\right] + s_{m1} \sin\left(\omega - \pi \operatorname{int}\left(\frac{\omega}{\pi}\right)\right)}{|s_{m2}| \sin\left(\omega - \pi \operatorname{int}\left(\frac{\omega}{\pi}\right)\right)} \right) \quad (3.130)$$

$$\rho_m(\omega, s_m) = \frac{1}{\sqrt{2}} \left\{ |s_{m2}|^2 + s_{m1} \sin(2\omega) - [|s_m|^2 - 1] \cos(2\omega) \right\}^{\frac{1}{4}} \quad (3.131)$$

### 3.7.10 Dislocation enrichment

Earlier simulations used a simple Heaviside function to represent the glide plane of edge dislocations. Such a model, in fact, somehow ignores the effect of dislocation core. Alternative functions are the compatible form (Fig. 3.20),

$$\Psi_H^\alpha = \mathbf{b}^\alpha \sum_{h=1}^{mh} N_h(\mathbf{x}) \left[ H(f^\alpha(\mathbf{x})) - H(f^\alpha(\mathbf{x}_I)) \right] H(g^\alpha(\mathbf{x})) \quad (3.132)$$

and the tapered Somigliana dislocation proposed by Gracie *et al.* (2007) (Fig. 3.20),

$$\Psi_H^\alpha = \mathbf{b}^\alpha \sum_{h=1}^{mh} N_h(\mathbf{x}) \left[ H(f^\alpha(\mathbf{x})) - H(f^\alpha(\mathbf{x}_I)) \right] \psi_H(g^\alpha(\mathbf{x})) \quad (3.133)$$

$$\psi_H(g(\mathbf{x})) = \begin{cases} 1 & g(\mathbf{x}) > 0 \\ \frac{(a^m - g(\mathbf{x})^m)^n}{a^{mn}} & -a < g(\mathbf{x}) < 0 \\ 0 & g(\mathbf{x}) < -a \end{cases} \quad (3.134)$$

where parameter  $n$ ,  $m$  and  $a$  are material properties which have to be obtained from experimental results or atomistic simulations.

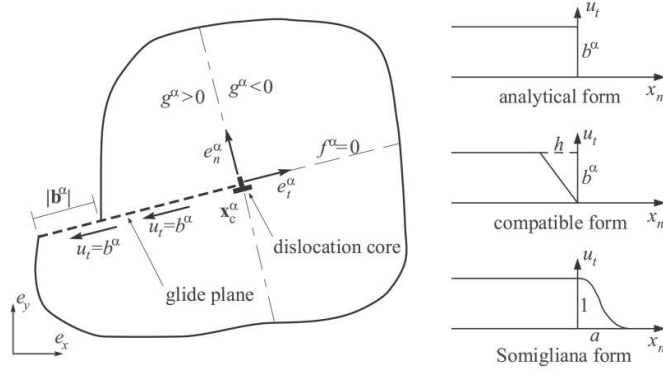


**Figure 3.20** Definition of functions  $f(\mathbf{x})$  and  $g(\mathbf{x})$  for an edge dislocation, and the incompatible, linearly regularized and tapered Somigiana profiles for a dislocation glide plane.

It is important to note that dissimilar to conventional XFEM solutions, no additional degrees of freedom are introduced in XFEM approximation due to the fact that the problem is solved for known prescribed sliding  $\mathbf{b}^\alpha = b_\alpha \mathbf{e}_t^\alpha$  along the sliding plane (Malekafzali, 2010 and Mohammadi and Malekafzali, 2011).

### 3.7.11 Hydraulic fracture enrichment

Crack tip enrichment functions for a hydraulic fracture problem can be defined as (Lecampion, 2009, Goodarzi *et al.*, 2011),

$$\{F_l(r, \theta)\}_{l=1}^4 = \{r^\lambda \cos(\lambda\theta), r^\lambda \sin(\lambda\theta), r^\lambda \sin\theta \sin(\lambda\theta), r^\lambda \sin\theta \cos(\lambda\theta)\} \quad (3.135)$$

where  $\lambda$  is the power of singularity, determined from  $K_m$  which includes the effects of solid material modulus  $E'$ , defined in (2.113), fracture toughness  $K_c$ , fluid viscosity  $\mu$  and crack propagation velocity  $v_c$ ,

$$K_m = \frac{K_c}{\sqrt{12\mu E' v_c}} \quad (3.136)$$

For large values of  $K_m$ , the problem is toughness dominated, (Lecampion, 2009)

$$\lambda = \frac{1}{2} \quad (3.137)$$

and the fluid pressure  $p$  crack can be represented by ,

$$p^h(\mathbf{x}) = \sum_{j=1}^n N_j(\mathbf{x}) p_j + \sum_{k=1}^{mf} N_k(\mathbf{x}) \left( \sum_{l=1}^{mf} F_l(x) \mathbf{b}_k^l \right) \quad (3.138)$$

$$\{F_l(r, \theta)\}_{l=1}^4 = \{0, H(\mathbf{x} - \mathbf{x}_{tip}), 0, 0\} \quad (3.139)$$

In contrast, the hydraulic fracture problem is viscosity controlled for low values of  $K_m$  (Lecampion, 2009),

$$\lambda = \frac{2}{3} \quad (3.140)$$

and

$$\{F_l(r, \theta)\}_{l=1}^4 = \left\{ 0, (\mathbf{x} - \mathbf{x}_{tip})^{-\frac{1}{3}} H(\mathbf{x} - \mathbf{x}_{tip}), 0, 0 \right\} \quad (3.141)$$

### 3.7.12 Plastic enrichment

Elguedj *et al.* (2006) proposed the concept of plastic enrichment to include effects of crack tip plasticification based on the Ramberg–Osgood plasticity model. In principle, other plasticity models may also be adopted with some modifications. Beginning with the Hutchinson–Rice–Rosengren power law hardening material model (Hutchinson, 1968):

$$\frac{\varepsilon}{\varepsilon_{\text{yld}}} = \frac{\sigma}{\sigma_{\text{yld}}} + k_0 \left( \frac{\sigma}{\sigma_{\text{yld}}} \right)^n \quad (3.142)$$

Asymptotic crack tip displacement, strain and stress fields can be defined as (Elguedj *et al.*, 2006):

$$u_i = k_0 \varepsilon_{\text{yld}} r \left( \frac{J}{k_0 \sigma_{\text{yld}} \varepsilon_{\text{yld}} I_n r} \right)^{\frac{n}{n+1}} \bar{u}_i(\theta, n) \quad (3.143)$$

$$\varepsilon_{ij} = k_0 \varepsilon_{\text{yld}} \left( \frac{J}{k_0 \sigma_{\text{yld}} \varepsilon_{\text{yld}} I_n r} \right)^{\frac{n}{n+1}} \bar{\varepsilon}_{ij}(\theta, n) \quad (3.144)$$

$$\sigma_{ij} = \sigma_{\text{yld}} \left( \frac{J}{k_0 \sigma_{\text{yld}} \varepsilon_{\text{yld}} I_n r} \right)^{\frac{1}{n+1}} \bar{\sigma}_{ij}(\theta, n) \quad (3.145)$$

where  $J$  is the well-known contour integral and  $I_n$  is a dimensionless constant that depends on  $n$ . The terms  $\bar{\sigma}_{ij}$ ,  $\bar{\varepsilon}_{ij}$  and  $\bar{u}_{ij}$  are dimensionless angular functions. Applying a Fourier decomposition on  $\bar{u}_{ij}$  for modes I and II allows for the pure mode I and II displacement fields to be defined from the following basis function

$$r^{\frac{1}{n+1}} \left\{ \left( \cos \frac{k\theta}{2}, \sin \frac{k\theta}{2} \right); k \in [1, 3, 5, 7] \right\} \quad (3.146)$$

In practice, Elguedj *et al.* (2006) have used and compared the following options derived from Eq. (3.146):

$$r^{\frac{1}{n+1}} \left\{ \sin \frac{\theta}{2}, \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \sin \theta, \cos \frac{\theta}{2} \sin \theta, \sin \frac{\theta}{2} \sin 2\theta, \cos \frac{\theta}{2} \sin 2\theta \right\} \quad (3.147)$$

$$r^{\frac{1}{n+1}} \left\{ \sin \frac{\theta}{2}, \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \sin \theta, \cos \frac{\theta}{2} \sin \theta, \sin \frac{\theta}{2} \sin 3\theta, \cos \frac{\theta}{2} \sin 3\theta \right\} \quad (3.148)$$

$$r^{\frac{1}{n+1}} \left\{ \sin \frac{\theta}{2}, \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \sin \theta, \cos \frac{\theta}{2} \sin \theta, \right. \\ \left. \sin \frac{\theta}{2} \sin 2\theta, \cos \frac{\theta}{2} \sin 2\theta, \sin \frac{\theta}{2} \sin 3\theta, \cos \frac{\theta}{2} \sin 3\theta \right\} \quad (3.149)$$

### 3.7.13 Viscoelastic enrichment

The correspondence theory has been frequently used to co-relate the viscoelastic response to its equivalent linear elastic behaviour. Accordingly, Zhang *et al.* (2010) and TianTang and QingWen (2011) have illustrated that the displacement fields around a crack tip within a viscoelastic domain can be written in the general forms of,

$$u_x = (1+\nu) I(t) K_I(t) \sqrt{\frac{r}{2\pi}} \left[ \kappa - 1 + 2 \sin^2 \frac{\theta}{2} \right] \cos \frac{\theta}{2} \\ + (1+\nu) I(t) K_{II}(t) \sqrt{\frac{r}{2\pi}} \left[ \kappa + 1 + 2 \cos^2 \frac{\theta}{2} \right] \sin \frac{\theta}{2} \quad (3.150)$$

$$u_x = (1+\nu) I(t) K_I(t) \sqrt{\frac{r}{2\pi}} \left[ \kappa + 1 - 2 \cos^2 \frac{\theta}{2} \right] \sin \frac{\theta}{2} \\ - (1+\nu) I(t) K_{II}(t) \sqrt{\frac{r}{2\pi}} \left[ \kappa - 1 - 2 \sin^2 \frac{\theta}{2} \right] \cos \frac{\theta}{2} \quad (3.151)$$

where  $I(t)$  is the corresponding creep compliance, determined from the viscoelastic material properties (Zhang, 2006).

As a result, despite the time dependency of the response and the stress intensity factors, the isotropic enrichment functions (3.95) can be used for radial and angular enhancement of the crack tip displacement approximation.

$$\{F_l(r, \theta)\}_{l=1}^4 = \left\{ \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \sin \theta \cos \frac{\theta}{2}, \sqrt{r} \sin \theta \sin \frac{\theta}{2} \right\} \quad (3.152)$$

### 3.7.14 Contact corner enrichment

Ebrahimi (2012) and Ebrihimi *et al.* (2012) developed an iterative approach to solve for the dominant singularity power  $\lambda$  at the corner of a contact wedge at various free/stick/slip contact boundary conditions (see Fig. 3.21).

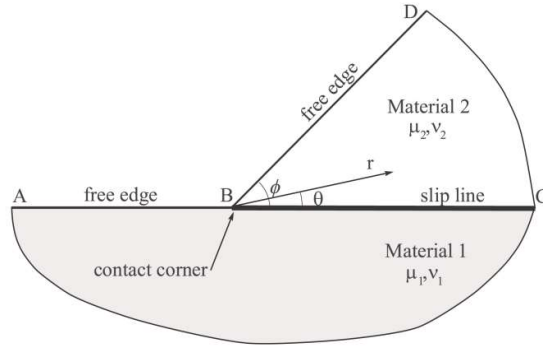


**Figure 3.21** Configuration of a complete sliding contact.

The singularity power  $\lambda$  for a complete sliding punch on a dissimilar plane surface can be obtained from the following characteristic equation (see Ebrahimi, 2012 and Ebrihimi *et al.*, 2012)

$$8(1 + \lambda) \sin \lambda \pi \left\{ (1 + \alpha) \cos \lambda \pi (\sin^2 \lambda \phi - \lambda^2 \sin^2 \phi) + \frac{1}{2} (1 - \alpha) \sin \lambda \pi (\sin 2\lambda \phi + \lambda \sin 2\phi) + \mu \sin \lambda \pi \left[ (1 - \alpha) \lambda (1 + \lambda) \sin^2 \phi - 2\beta (\sin^2 \lambda \phi - \lambda^2 \sin^2 \phi) \right] \right\} = 0 \quad (3.153)$$

where  $\phi$  is the wedge angle and  $\alpha$  and  $\beta$  are the Dundurs parameters for dissimilar bimetals,

$$\alpha = \frac{(\mu_2 / \mu_1)(\kappa_1 + 1) - (\kappa_2 + 1)}{(\mu_2 / \mu_1)(\kappa_1 + 1) + (\kappa_2 + 1)} \quad (3.154)$$

$$\beta = \frac{(\mu_2 / \mu_1)(\kappa_1 - 1) - (\kappa_2 - 1)}{(\mu_2 / \mu_1)(\kappa_1 + 1) + (\kappa_2 + 1)} \quad (3.155)$$

Ebrahimi (2012) and Ebrihimi *et al.* (2012) developed a local Newton's iterative method to solve for the dominant  $\lambda$ . The contact corner enrichment functions are then defined as,

$$\{F_l(r, \theta)\}_{l=1}^4 = \{r^\lambda \cos(\lambda\theta), r^\lambda \sin(\lambda\theta), r^\lambda \sin\theta \sin(\lambda\theta), r^\lambda \sin\theta \cos(\lambda\theta)\} \quad (3.156)$$

### 3.7.15 Modification for large deformation problems

Bouchbinder *et al.* (2009 and 2010) have computed the complete weakly nonlinear quasi-static solution for the displacement field for neo-Hookean materials,

$$u_x = \frac{K_I}{4\mu} \sqrt{\frac{r}{2\pi}} \left[ \frac{7}{3} \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \right] + \left( \frac{K_I}{4\mu\sqrt{2\pi}} \right)^2 \left[ -\frac{1}{15} \ln r - \frac{52}{45} \left( \ln r + \frac{3}{4} \sin^2 \theta \right) - \frac{103}{48} \cos \theta + \frac{26}{15} \cos 2\theta - \frac{3}{16} \cos 3\theta \right] \quad (3.157)$$

$$u_y = \frac{K_I}{4\mu} \sqrt{\frac{r}{2\pi}} \left[ \frac{13}{3} \sin \frac{\theta}{2} - \sin \frac{3\theta}{2} \right] + \left( \frac{K_I}{4\mu\sqrt{2\pi}} \right)^2 \left[ \frac{\theta}{15} - \frac{52}{45} \left( \frac{\theta}{4} - \frac{3}{8} \sin 2\theta \right) - \frac{61}{48} \sin \theta + \frac{26}{15} \sin 2\theta - \frac{3}{16} \sin 3\theta \right] \quad (3.158)$$

As a result, in addition to the term  $\sqrt{r}$ , a new term  $\ln r$  can be included in the enrichment function.

In a total Lagrangian formulation for large deformation analysis, the basic XFEM stiffness formulation (3.43) is modified to (Rashetnia *et al.*, 2012):

$$(\mathbf{K}_{ab})_{ij} = \sum_{I,J,K,L=1}^3 \int_{\Omega^e} \mathbf{F}_{iI} N_{a,J} \mathbf{D}_{IJKL} N_{b,K} \mathbf{F}_{jL} d\Omega, \quad (r, s = \mathbf{u}, \mathbf{a}, \mathbf{b}) \quad (3.159)$$

or in a more appropriate form of

$$\mathbf{K}_{ij}^{rs} = \int_{\Omega^e} (\mathbf{B}_{0a}^r)^T \mathbf{D} \mathbf{B}_{0b}^s d\Omega \quad (3.160)$$

where  $\mathbf{B}_0$  is defined in terms of the derivatives of the shape functions with respect to the initial configuration  $\mathbf{X} = (X, Y)$  and the components of deformation gradient  $\mathbf{F}$  (Rashetnia *et al.*, 2012),

$$\mathbf{B}_{0i}^u = \begin{bmatrix} N_{i,X} \mathbf{F}_{11} & N_{i,X} \mathbf{F}_{21} \\ N_{i,Y} \mathbf{F}_{11} & N_{i,Y} \mathbf{F}_{22} \\ N_{i,X} \mathbf{F}_{12} + N_{i,Y} \mathbf{F}_{11} & N_{i,X} \mathbf{F}_{22} + N_{i,Y} \mathbf{F}_{21} \end{bmatrix} \quad (3.161)$$

$$\mathbf{B}_i^a = \begin{bmatrix} N_{i,X} \mathbf{F}_{11} H & N_{i,X} \mathbf{F}_{21} H \\ N_{i,Y} \mathbf{F}_{11} H & N_{i,Y} \mathbf{F}_{22} H \\ [N_{i,X} \mathbf{F}_{12} + N_{i,Y} \mathbf{F}_{11}] H & [N_{i,X} \mathbf{F}_{22} + N_{i,Y} \mathbf{F}_{21}] H \end{bmatrix} \quad (3.162)$$

$$\mathbf{B}_i^b = [\mathbf{B}_i^{b1} \quad \mathbf{B}_i^{b2} \quad \mathbf{B}_i^{b3} \quad \mathbf{B}_i^{b4}] \quad (3.163)$$

$$\mathbf{B}_i^{b\alpha} = \begin{bmatrix} (N_i F_\alpha)_{,X} \mathbf{F}_{11} & (N_i F_\alpha)_{,X} \mathbf{F}_{21} \\ (N_i F_\alpha)_{,Y} \mathbf{F}_{12} & (N_i F_\alpha)_{,Y} \mathbf{F}_{22} \\ (N_i F_\alpha)_{,X} \mathbf{F}_{12} + (N_i F_\alpha)_{,Y} \mathbf{F}_{11} & (N_i F_\alpha)_{,X} \mathbf{F}_{22} + (N_i F_\alpha)_{,Y} \mathbf{F}_{21} \end{bmatrix} \quad (3.164)$$

where the deformation gradient  $\mathbf{F}_{il}$  is defined as the derivative of current configuration  $\mathbf{x}$  with respect to the initial configuration  $\mathbf{X}$ ,

$$\mathbf{F}_{il} = \frac{\partial \mathbf{x}_i}{\partial \mathbf{X}_l} = \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix} \quad (3.165)$$

To include the effects of interpolation, the following shifting amendments are required:

$$\mathbf{B}_i^a = \begin{bmatrix} N_{i,X} \mathbf{F}_{11} \bar{H}_i & N_{i,X} \mathbf{F}_{21} \bar{H}_i \\ N_{i,Y} \mathbf{F}_{11} \bar{H}_i & N_{i,Y} \mathbf{F}_{22} \bar{H}_i \\ [N_{i,X} \mathbf{F}_{12} + N_{i,Y} \mathbf{F}_{11}] \bar{H}_i & [N_{i,X} \mathbf{F}_{22} + N_{i,Y} \mathbf{F}_{21}] \bar{H}_i \end{bmatrix} \quad (3.166)$$

$$\mathbf{B}_i^{b\alpha} = \begin{bmatrix} (N_i \bar{F}_{\alpha i})_{,X} \mathbf{F}_{11} & (N_i \bar{F}_{\alpha i})_{,X} \mathbf{F}_{21} \\ (N_i \bar{F}_{\alpha i})_{,Y} \mathbf{F}_{12} & (N_i \bar{F}_{\alpha i})_{,Y} \mathbf{F}_{22} \\ (N_i \bar{F}_{\alpha i})_{,X} \mathbf{F}_{12} + (N_i \bar{F}_{\alpha i})_{,Y} \mathbf{F}_{11} & (N_i \bar{F}_{\alpha i})_{,X} \mathbf{F}_{22} + (N_i \bar{F}_{\alpha i})_{,Y} \mathbf{F}_{21} \end{bmatrix} \quad (3.167)$$

where

$$\bar{H}_i = H(\xi) - H(\xi_i) \quad (3.168)$$

$$\bar{F}_{\alpha i} = F_\alpha - F_{\alpha i}, \quad \alpha = 1 - 4 \quad (3.169)$$